Economics 300-01: Quantitative Methods in Economics
Wesleyan University, Fall 2007
Selected Answers to Problem Set #2

3-14: The key to this problem is to recognize that each attempted sale is viewed as independent of the others — that is, the probability on each is 20%, and does not vary if the salesperson is more or less successful on other attempts.

1. Since the sales are independent, the probability of *not* making a single sale is 80%. Then, the probability of not making *any* sales in 12 attempts is

$$0.8 \times 0.8 \times \cdots \times 0.8 = (0.8)^{12} = 0.0687 \approx 7\%.$$

   (Notice how this is an application of the binomial distribution introduced in chapter 4.)

2. Since the probability of making at least one sale is the complement of not making any sales, this probability is approximately $1 - 0.07 = 0.93$, or 93%.

3. Saying the salesperson has roughly a 93% chance of making a sale on any one day is analogous to saying that the salesperson makes at least one sale 93% of the time. Therefore, in a 200 day period, the salesperson should make at least one sale on $200 \times 0.93 = 186$ of those days. (Notice that this too is an application of the binomial distribution, although here $\pi = 0.93$ and $n = 200$, whereas $\pi = 0.8$ and $n = 12$ in part **a**.)

3-18: This problem is very much like the example we did on the board in class, with the mixture of red and black balls in an urn.

1. Initially, there is a 6 in 10 chance of drawing a good light bulb (event G_1; $\Pr(G_1) = \frac{6}{10}$). The probability of drawing two good bulbs can be found via the rule for conditional probability: it is $\Pr(G_1 \cap G_2) = \Pr(G_2|G_1) \cdot \Pr(G_1)$. Since the first light bulb is not replaced, $\Pr(G_2|G_1) = \frac{5}{9}$. Thus, $\Pr(G_1 \cap G_2) = \left(\frac{6}{10}\right) \cdot \left(\frac{5}{9}\right) = \frac{1}{3}$. (Notice how it is usually easier to solve this type of question with fractions instead of decimals.)

2. Continuing this line of reasoning, the probability that the third is good is $\frac{4}{8}$: of the eight remaining bulbs, four are good, so you have a 4 in 8 chance of picking one of them. Symbolically, $Pr(G_3|G_1 \cap G_2) = \frac{4}{8} = \frac{1}{2}$. Then, not replacing any of the first three bulbs, the probability of choosing a fourth good bulb is $\frac{3}{7}$. And so the probability of choosing a fifth good bulb — conditional on choosing good bulbs for the first four — is $\frac{2}{6}$. Now the question asks the probability that the next three bulbs are good, given the first two are good. It is simply the product of the three fractions in the preceding sentences of this part: $\left(\frac{4}{8}\right) \cdot \left(\frac{3}{7}\right) \cdot \left(\frac{2}{6}\right) = \frac{1}{14} \approx 7\%$. In symbols, this can be written as $\Pr(G_5 \cap G_4 \cap G_3 | G_1 \cap G_2)$. We could write out the tree diagram for this problem (it would be rather large), but hopefully the logic of the problem is clear enough.

3. If we start all over again, we will need to calculate the probability of choosing 5 good bulbs without replacement: $\Pr(G_5 \cap G_4 \cap G_3 \cap G_2 \cap G_1)$. While this may seem imposing, it is not — and we already have all the pieces for the answer! Recall the rule for conditional probability is $\Pr(A \cap B) = Pr(B|A) \cdot Pr(A)$.

Now define one compound event $A = G_2 \cap G_1$, and a second compound event $B = G_5 \cap G_4 \cap G_3$. Applying the above formula, we see that:

$$\begin{aligned}
\Pr(G_5 \cap G_4 \cap G_3 \cap G_2 \cap G_1) &= \Pr(A \cap B) \\
&= \Pr(B|A) \cdot \Pr(A) \\
&= \Pr(G_5 \cap G_4 \cap G_3 | G_2 \cap G_1) \cdot \Pr(G_2 \cap G_1) \\
&= \left(\frac{1}{14}\right) \cdot \left(\frac{1}{3}\right) \\
&= \frac{1}{42} \approx 2.4\%.
\end{aligned}$$

While this might seem like a “trick,” it is nothing of the sort! Rather, it is a straight-forward (if not obvious) application of the definitions of conditional probability and compound events. If you try divide the compound event $G_5 \cap G_4 \cap G_3 \cap G_2 \cap G_1$ in a different way, and calculate the corresponding probabilities, you should still get the same answer. (Try it!)

A number of problems in statistics can be solved by finding convenient (and occasionally clever) ways to re-write the problem such that more simple manipulation and/or calculation results. This is a skill that can be learned, but it takes a fair bit of practice!

3-38: The easiest way to answer this question is to draw a tree diagram. Below I provide intuitive and mathematical answers; you should check that they match your tree diagram and come see me if you have any questions.

Let’s start by figuring out what information is given to us. We’ll use the following notation:

$$\begin{aligned}
T &= \text{“guilty” on lie detector test} \\
\overline{T} &= \text{“innocent” on lie detector test} \\
S &= \text{stole from the company} \\
\overline{S} &= \text{did not steal from the company}
\end{aligned}$$

Then, we know that $\Pr(T|S) = 0.9$ and $\Pr(\overline{T}|\overline{S}) = 0.9$ as well. That is, given that we knew whether past workers were guilty or not, the lie detector test correctly determined their innocence or guilt 90% of the time in the past. We also are told that $\Pr(S) = 0.05$.

1. If a worker is fired, he or she must have failed the test, so we condition on the event T. The probability of being innocent but failing the test can be found through Bayes' Rule:

$$\begin{aligned}
\Pr(\overline{S}|T) &= \frac{\Pr(\overline{S}\cap T)}{\Pr(T)} \\
&= \frac{\Pr(\overline{S}\cap T)}{\Pr(\overline{S}\cap T)+\Pr(S\cap T)} \\
&= \frac{\Pr(T|\overline{S})\Pr(\overline{S})}{\Pr(T|\overline{S})\Pr(\overline{S})+\Pr(T|S)\Pr(S)} \\
&= \frac{(0.1)(0.95)}{(0.1)(0.95)+(0.9)(0.05)} \\
&= \frac{0.095}{0.095+0.045} \approx 0.679\,.
\end{aligned}$$

   Thus, 68% of those fired were innocent! What happened? Looking closely at the values in the numerator and denominator should clarify the issue: most people are innocent — in fact, 95% of them. But 10% of these innocent people will nonetheless fail the lie detector test. Hence if the firm started with 1000 workers, 95 innocent workers would be fired. On the other hand, of the 5% of the workers who were guilty, 90% were "found out" and fired. Thinking again of an initial 1000 workers, only 50 stole and 45 of them were fired. Since so few people stole in the first place, and the since the lie detector test is not especially accurate, many more innocent people in total would be fired than guilty people. Thus, as a share of the total group of people fired, the proportion who are innocent is relatively large.

2. For those workers not fired (i.e. event $\overline{T}$), we can calculate the proportion who actually are guilty:

$$\begin{aligned}
\Pr(S|\overline{T}) &= \frac{\Pr(S\cap \overline{T})}{\Pr(\overline{T})} \\
&= \frac{\Pr(S\cap \overline{T})}{\Pr(S\cap \overline{T})+\Pr(\overline{S}\cap \overline{T})} \\
&= \frac{\Pr(\overline{T}|S)\Pr(S)}{\Pr(\overline{T}|S)\Pr(S)+\Pr(\overline{T}|\overline{S})\Pr(\overline{S})} \\
&= \frac{(0.1)(0.05)}{(0.1)(0.05)+(0.9)(0.95)} \\
&= \frac{0.005}{0.005+0.855} \approx 0.0058\,.
\end{aligned}$$

   In other words, less than 1% of those not fired have stolen from the company. So most of the guilty people have in fact left the firm, but at the cost of losing a large number of innocent people as well — not to mention possibly risking some law suits.

3-42: It might be easiest to answer this question by drawing a probability tree as well. Here is one version of what it might look like: consider the three men each randomly given a hat in sequential order. The first man thus has a $\frac{1}{3}$ chance of getting his own hat, and a $\frac{2}{3}$ chance of getting one of the other two. If the first man actually gets his own hat, then of the two remaining hats there is a $\frac{1}{2}$ chance the other two will also get their hats (which we shall denote as "case 1"), and a $\frac{1}{2}$ chance they will get each other's hat ("case 2").

In the lower branch, if the first man does not get his hat, then at least one other person also will not get his hat (because the first man is holding it). So there is a $\frac{1}{2}$ chance that one of the other two men will in fact still get his hat ("case 3"), and a $\frac{1}{2}$ chance that *none* of the men end up with their respective hats ("case 4").

Notice that the correctly drawn tree has only two "levels," not three: once the first two hats have been allocated, there is no third choice (third set of branches) for the final individual. He simply gets the hat that is remaining.

1. The probability that no man gets his hat is case 4, along the lower branch. The probability of this occurring is $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)=\frac{1}{3}$.

2. There are two cases in which exactly one man gets the right hat. One is case 2, in which only the first man gets his hat, which occurs with probability $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)=\frac{1}{6}$. The other is case 3, in which one of the two remaining individuals gets his hat. This event occurs with probability $\left(\frac{2}{3}\right)\left(\frac{1}{2}\right)=\frac{1}{3}$. Thus, the chance that exactly one man ends up with his own hat is $\frac{1}{6}+\frac{1}{3}=\frac{1}{2}$.

3. The probability that exactly two men will end up with the correct hats is, intuitively, zero. There are only three hats: if two men have their hats then so must the third, and if one man does not have the right hat then another is holding this person's hat — which means a second must not have the right hat either. So this event is the null set, and has probability zero.

4. All three men end up with the right hat in case 1, which occurs with probability $\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)=\frac{1}{6}$.

Notice that $\frac{1}{3}+\frac{1}{2}+0+\frac{1}{6}=1$, so we have defined probabilities for the full sample space.

Revised: 21-Sep-2007